

Minnesota State High School Mathematics League

Individual Event

2004-05 Event 5A

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 20 minutes for this event.

18

1. If the integers 1, 2, ..., 8 are entered into the eight squares in Figure 1 so that successive integers are not in squares that are adjacent or have a common corner, then what is the sum of the four integers in the middle column?

50

2. 99% of the weight of 100 pounds of potatoes is water. An overnight drying process reduces the amount of water so it constitutes only 98% of the weight of the potatoes. What do the potatoes then weigh?

3. New year's Day in 2005 was on a Saturday. The grid in Figure 3 is torn from a January, 2005 calendar. The dates in the marked squares sum to 50. What three days of the week are represented in the part of the calendar shown?

A = Tues
B = Wed
C = Thurs

4. Racing to see who can be the first to swim 10 laps, Allison and Beth start at opposite ends of a pool. Each girl swims steadily at her own rate. They first meet at a point a yards from the side where Allison began, and they next meet b yards from the side where Beth began. What, in terms of a and b , is the length of the pool?

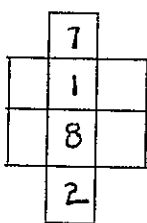
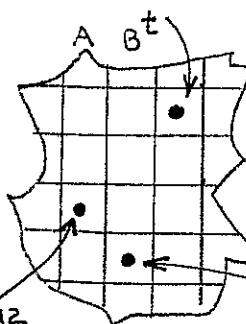


Figure 1

1. Since each of the two middle squares touch six others, they must contain 1 and 8 (in some order). This fixes locations for 2 and 7. That's really all we need.



3. Let the date in the upper right be t . Then the others are as marked, and

$$t + (t+12) + (t+20) = 50$$

$$3t = 18; t = 6. \text{ Since}$$

the first was on Saturday, so also is the 8th. The sixth is Thurs.

2. The bulk in the potatoes weighs 1 pound. Let W be the weight of the potatoes after drying (when they are 98% water).

$$.02W = 1; W = \frac{1}{.02} = 50$$

L = length of pool.

4. A swims x yds/min

B swims y yds/min

When they first meet at time t ,

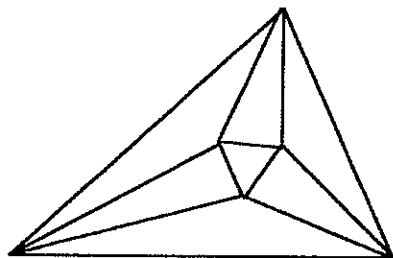
$$\begin{cases} tx = a \\ ty = L - a \end{cases} \Rightarrow \frac{x}{y} = \frac{a}{L-a}$$

When they next meet at time T ,

$$\begin{cases} Tx = L + b \\ Ty = 2L - b \end{cases} \Rightarrow \frac{x}{y} = \frac{L+b}{2L-b}$$

$$\text{Solve } \frac{a}{L-a} = \frac{L+b}{2L-b}$$

$$L = 3a - b$$



Minnesota State High School Mathematics League

Individual Event

2004-05 Event 5B

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

54

1. Find the area of the trapezoid (Figure 1) having bases of 5 and 13, and a height of 6.

1600

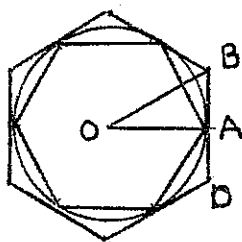
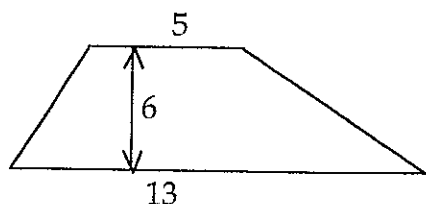
2. Using the tape used to cordon off an area, police need 3 rolls to enclose a square containing 900 square yards. What (in square yards) will be the area of a square that can be enclosed using 4 rolls?

 $\frac{4}{3}$

3. A regular hexagon with area I is inscribed in a circle. At each vertex, a line segment tangent to the circle is drawn. These line segments form a second regular hexagon with area C circumscribing the circle. Find the ratio $\frac{C}{I}$.

Accept $\frac{\pi h}{3} \cdot \frac{R^3 - r^3}{R - r}$

4. The base of a right circular cone has radius R . The top of the cone is cut off by a plane parallel to and h units above the plane of the base. The resulting solid, called the frustum of a cone, has a circular top of radius $r < R$. Express the volume of a frustum of a cone in terms of r , R and h .



4. Volume of the large cone $V_L = \frac{\pi}{3} R^2 s$; volume of the removed cone $V_S = \frac{\pi}{3} r^2 t$.

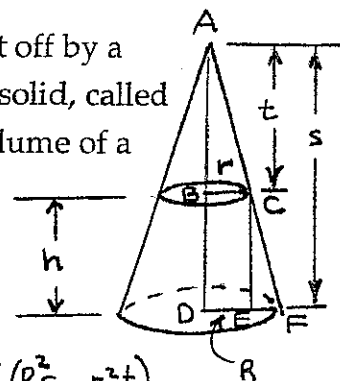
Desired volume $= V = V_L - V_S = \frac{\pi}{3} (R^2 s - r^2 t)$

$\triangle ABC \sim \triangle CEF$ so $\frac{t}{r} = \frac{h}{R-r}$; $t = \frac{rh}{R-r}$

$\triangle ADF \sim \triangle CEF$ so $\frac{s}{R} = \frac{h}{R-r}$; $s = \frac{Rh}{R-r}$

$$V = \frac{\pi}{3} \left[R^2 \frac{Rh}{R-r} - r^2 \frac{rh}{R-r} \right] = \frac{\pi h}{3(R-r)} [R^3 - r^3]$$

$$= \frac{\pi h}{3} [R^2 + Rr + r^2]$$

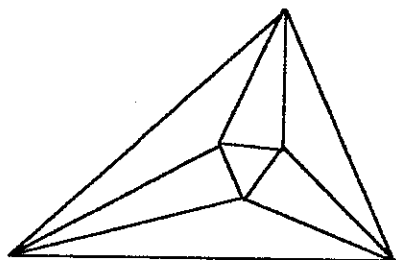


1. Area $= h \cdot \frac{\text{sum of bases}}{2} = 6 \cdot \frac{13+5}{2} = 54$

2. Let r be the length in yards of one roll.
Then $\left(\frac{3r}{4}\right)^2 = 900$; $r^2 = 16(100)$; $r = 40$
One roll on each side encloses 1600

3. The area of a regular hexagon with side length a is $6 \cdot \frac{\sqrt{3}}{4} a^2 = \frac{3\sqrt{3}}{2} a^2$
Let $t = BD$ so $C = \frac{3\sqrt{3}}{2} t^2$

Then $AB = \frac{t}{2}$ so $OA = \frac{t}{2} \sqrt{3}$ and $I = \frac{3\sqrt{3}}{2} \left(\frac{t}{2} \sqrt{3}\right)^2$
 $\therefore \frac{C}{I} = \frac{t^2}{\frac{t^2}{4} (3)} = \frac{4}{3}$



Minnesota State High School Mathematics League

Individual Event

2004-05 Event 5C

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

- 1/6 1. Two ordinary dice are rolled. What is the probability that the total showing on the dice will be 10 or greater?
- 35 2. From 7 applicants for a job, the personnel department of a company is to present to management in alphabetical order a list of 3 suitable candidates. How many different lists are possible?
- 80 3. How many even three digit numbers can be formed using digits from the set $\{0,1,2,3,5,6,7\}$ if all digits of the integer must be distinct?
- 53,130 4. In how many ways can 20 identical balls be distributed into six cans labeled A, B, C, D, E, F? Three ways, for example, would be $\{5,5,4,3,2,1\}$, $\{4,5,5,3,2,1\}$ and $\{9,5,0,1,2,3\}$.

1. Successes are

5 5
5 6
6 5 prob = $\frac{6}{36}$
6 6
6 4
4 6

3. Let the number be abc

Case I: a even, b odd $2 \cdot 4 \cdot 2 = 16$ ways
Case II: a even, b even $2 \cdot 2 \cdot 1 = 4$
Case III: a odd, b even $4 \cdot 3 \cdot 2 = 24$
Case IV: a odd, b odd $4 \cdot 3 \cdot 3 = 36$
80

2. After 3 are chosen, alphabetize. Thus, this is just

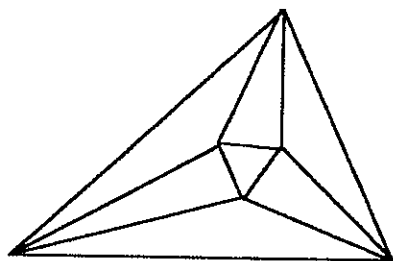
$$\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} = 35$$

4. Represent the three examples with a series of o's and vertical lines as follows:

A B C D E F
ooooo|ooooo|ooooo|ooooo|ooooo
ooooo|ooooo|ooooo|ooooo|ooooo
ooooooooo|ooooo|ooooo|ooooo|ooooo

In the same way, using 20 o's and 5 vertical lines, any partition (that's the technical name) of 20 into 6 classes can be represented. But 25 objects, with 20 alike and 5 alike can be arranged in

$$\frac{25!}{(20!)(5!)} = 53,130 \text{ ways}$$



Minnesota State High School Mathematics League

Individual Event

2004-05 Event 5D

Questions in this event are written, with permission, as variations of problems from the 2004 AMC-12 Exam. Review of this exam is excellent preparation, not only for this event, but for the 2005 AMC-12, which we strongly encourage you to take.

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

- 2005 1. What will be the result if the sum of the first 2004 even counting numbers (that is, the first 2004 even positive integers) is subtracted from the sum of the first 2005 odd counting numbers?

- 80 2. What is the median of 125 consecutive integers if their sum is 10,000?

- $\frac{3}{1}$ 3. Rectangle ABCD (Figure 3) has sides of length $AB = 4$, $BC = 7$. A semi-circle is constructed inside the rectangle using AB as a diameter, and a line drawn from C tangent to the semi-circle intersects AD at E . What is the length of CE ?

or $7\frac{4}{7}$ or 7.571

- $3 + \sqrt{7}$ 4. Figure 4 shows a view from the top of four spheres, each with radius 1, resting on a horizontal plane P so that their centers form a square of side length 2. Suppose a sphere of radius 2 is placed on top of them so it is tangent to all four of the smaller spheres. What is the distance from the plane P to the top of the larger sphere?

$$1. [2(1)-1] + [2(2)-1] + \dots + [2(2005)-1] = 1 + 3 + \dots + 4007 + 4009$$

$$- 2[1 + 2 + \dots + 2004] = -[2 + 4 + \dots + 4008]$$

$$\underbrace{-1 -1 \dots -1}_{2004 \text{ terms}} + 4009 = 4009 - 2004 = 2005$$

2. For 125 consecutive integers, the median = the mean = $\frac{\text{sum of integers}}{125} = \frac{10,000}{125} = 80$

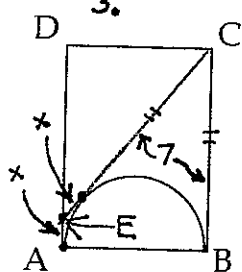


Figure 3

$$DE^2 + DC^2 = CE^2$$

$$(7-x)^2 + 4^2 = (7+x)^2$$

$$16 = 28x; x = \frac{4}{7}$$

$$CE = 7 + \frac{4}{7} = \frac{53}{7}$$

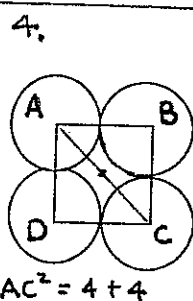
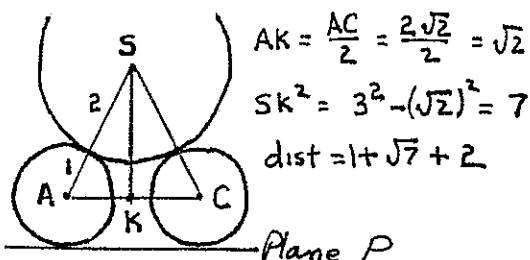
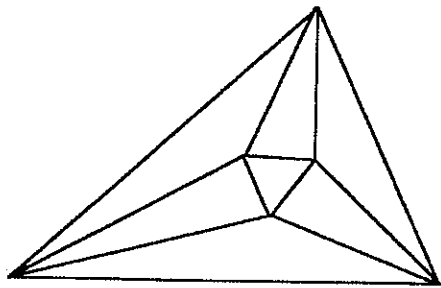


Figure 4

Side view showing spheres centered at A and C





Minnesota State High School Mathematics League

Team Event

2004-05 Meet 5

Each question is worth 4 points. Team members may cooperate in any way, but at the end of twenty minutes, one set of answers is to be submitted. Put answers on the lines provided.

1. Let A, B, C , and D be four points on a line with $AB = CD$ (Figure 1). Semi-circles are drawn above the line with diameters $AD = 2R$ and $BC = 2r$, and semi-circles with diameters of AB and CD are drawn below the line. If $R = 2m + 1$ and $r = 2n + 1$ are both odd integers, then the area of the region enclosed by the four semi-circles is an integral multiple of π . Express this integer in terms of m and n .

$$(3m + n + 2)(m - n) \text{ or } 3m^2 - 2mn + 2m - n^2 - 2n$$

- 12 2. In Figure 2, $AE \perp AB$, $BC \perp AB$, $AB = 6$, $BC = 7$, and $AE = 11$. Find $\text{Area}(\triangle ADE) - \text{Area}(\triangle BCD)$.

3. Let the arbitrary $\triangle ABC$ with sides a, b , and c (Figure 3) have area T . On each side of the triangle, construct an equilateral triangle having, respectively, areas

T_a, T_b , and T_c . If $\angle A = \frac{\pi}{3}$, a simple equation relates T, T_a, T_b , and T_c . Find it.

$$T = T_b + T_c - T_a$$

- $\frac{57}{14} = 1.929$ 4. A sudden rainstorm forced us to abandon our backyard picnic. When we came out the next morning, we found that an empty salad bowl, left behind in our dash to the house, had 4.5 inches of rainwater in it. Wondering how many inches of rain had fallen, we noted that the bowl was shaped like the frustum of a right circular cone, having a base with a diameter of 6 inches, a top diameter of 14 inches, and a depth of 6 inches. How many inches of rain fell; i.e. how many inches would have stood in a right circular cylinder if there had been one on the table?

510

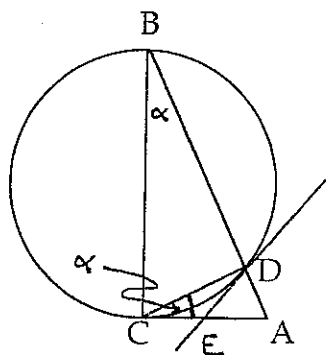
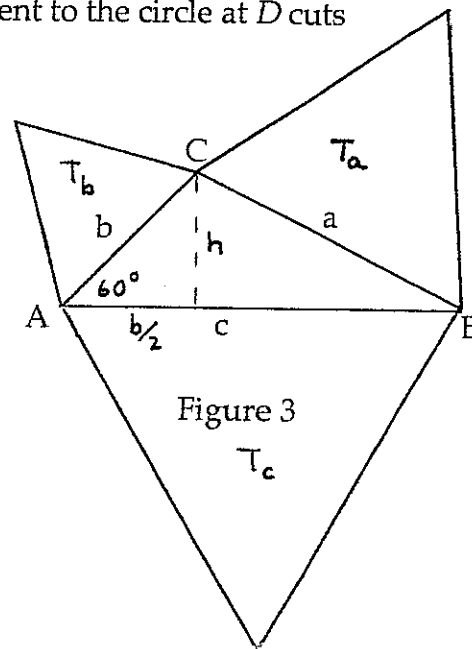
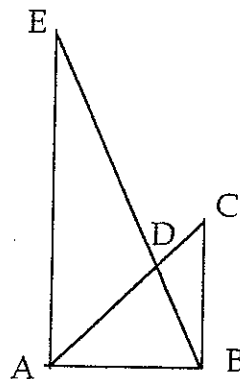
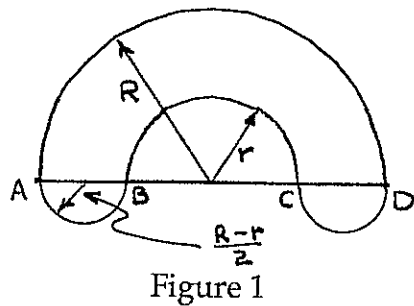
5. Starting at diametrically opposite points, Allison and Beth run in opposite directions on a circular track, each girl running steadily at her own rate. They first meet after Allison has run 150 yards. They next meet after Beth has run 210 yards past their first meeting point. What is the length of the track in yards?

Team _____

Answers

3/2

6. Right $\triangle ABC$ has legs of lengths 3 and 8. The long leg is used as the diameter of a circle that cuts the hypotenuse at D (Figure 6). A line tangent to the circle at D cuts AC at E . What is the length of CE ?



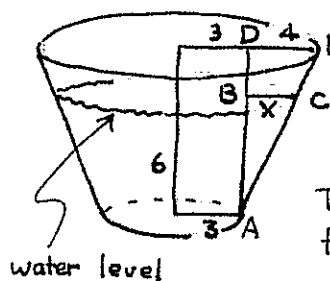
Team _____

Team Event 5 Solutions

1. See Figure 1 on the answer sheet.

$$\begin{aligned} \text{Area} &= \frac{1}{2} \pi R^2 - \frac{1}{2} \pi r^2 + \pi \left(\frac{R-r}{2} \right)^2 \\ &= \frac{\pi}{2} \left[R^2 - r^2 + \frac{1}{2} (R^2 - 2Rr + r^2) \right] \\ &= \frac{\pi}{4} (3R^2 - 2Rr - r^2) = \frac{\pi}{4} (3R+r)(R-r) \\ &= \frac{\pi}{4} [6m+3+2n+1] [(2m+1)-(2n+1)] \\ &= \frac{\pi}{4} [2(3m+n+2)] [2(m-n)] \end{aligned}$$

4. Water is collected by a circular surface of area $\pi(7)^2$. The question asks how high (in inches) water would stand in a cylinder with that base.



$$\Delta ABC \sim \Delta ADE$$

$$\frac{x}{4} = \frac{9/2}{6} = \frac{3}{4}$$

$$x = 3$$

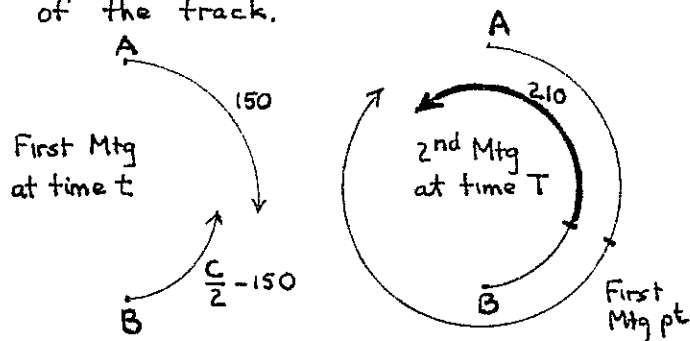
The water in the bowl fills the frustum of a cone that, according to problem 4 of

Event 5B has a volume of

$$\frac{\pi}{3} \cdot \frac{9}{2} [6^2 + 6 \cdot 3 + 3^2] = \frac{3\pi}{2} (63)$$

$$\therefore 49\pi h = \frac{3\pi}{2} (63); \quad h = \frac{27}{14} = 1.929$$

5. Let a = Allison's pace in yds/min, b = Beth's pace. Let their first meeting occur at time t , their second at T . Finally, let C be the circumference of the track.



$$at = 150$$

$$bt = \frac{C}{2} - 150$$

$$bT = \left(\frac{C}{2} - 150 \right) + 210 = \frac{C}{2} + 60$$

$$aT = C - \left[\frac{C}{2} + 60 - \frac{C}{2} \right] = C - 60$$

$$\frac{at}{bt} = \frac{150}{\frac{C}{2} - 150} = \frac{300}{C - 300}; \quad \frac{aT}{bT} = \frac{C - 60}{\frac{C}{2} + 60} = \frac{2C - 120}{C + 120}$$

$$\frac{aT}{bT} = \frac{C - 60}{\frac{C}{2} + 60} = \frac{2C - 120}{C + 120}$$

2. Area (ΔADE) - Area (ΔBCD)

$$\begin{aligned} &= [\text{Area}(\Delta ADE) + \text{Area}(\Delta ABD)] \\ &\quad - [\text{Area}(\Delta BCD) + \text{Area}(\Delta ABD)] \\ &= \left[\frac{1}{2} (6)(11) \right] - \left[\frac{1}{2} (6)(7) \right] = 12 \end{aligned}$$

3. See Figure 3 on answer sheet

$$T = \frac{1}{2} ch = \frac{1}{2} a \frac{b}{2} \sqrt{3} = \frac{1}{4} bc \sqrt{3}$$

$$T_a = \frac{a^2 \sqrt{3}}{4}; \quad T_b = \frac{b^2 \sqrt{3}}{4}; \quad T_c = \frac{c^2 \sqrt{3}}{4}$$

From the law of cosines,

$$a^2 = b^2 + c^2 - 2bc \cos\left(\frac{\pi}{3}\right)$$

$$\frac{4T_a}{\sqrt{3}} = \frac{4T_b}{\sqrt{3}} + \frac{4T_c}{\sqrt{3}} - 2\left(\frac{4T}{\sqrt{3}}\right)\left(\frac{1}{2}\right)$$

$$T_a = T_b + T_c - T$$

6. Let $\alpha = \angle ACD = \angle ABC$ (since these angles have \perp sides.

$EC = ED$ (tangents from same pt)

so ΔCDE is isosceles; $\alpha = \angle CDE$.

Let $\beta = \angle BCD = \angle CAB$, both complements of α . Then note that $CD \perp AB$ (for ΔCDB) is inscribed in a semicircle, so

we also have $\angle ADE$ as a complement to α ; $\angle ADE = \beta$

$\therefore \Delta ADE$ is isosceles.

Then $EA = ED = EC$

$$CE = \frac{1}{2} CA = \frac{3}{2}$$

$$\text{Solve } \frac{300}{C - 300} = \frac{2C - 120}{C + 120}$$

$$300C = 2C^2 - 720C$$

$$2C(C - 510) = 0$$

$$C = 510$$